Mean number of visits to sites in Levy flights

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Formulas are derived to compute the mean number of times a site has been visited during symmetric Levy flights. Unrestricted Levy flights are considered first, for lattices of any dimension: conditions for the existence of finite asymptotic maps of the visits over the lattice are analyzed and a connection is made with the transience of the flight. In particular it is shown that flights on lattices of dimension greater than 1 are always transient. For an interval with absorbing boundaries the mean number of visits reaches stationary values, which are computed by means of numerical and analytical methods; comparisons with Monte Carlo simulations are also presented.

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I. INTRODUCTION

Levy flights are a model of diffusion in which the probability of a $|z|$-length jump is "broad," in that, asymptotically, $p(z)\sim|z|^{-1-\alpha}$, $0<\alpha<2$. In this case the sum $x_k=\Sigma_i^k z_i$ is distributed according to a Levy distribution, whereas for $\alpha\geq 2$ normal diffusion takes place [1,2]. Interesting problems arise in the theory of Levy flights when considering the statistics of the visits to the sites, such as, for instance, the number of different sites visited during a flight [3,4]; in this Brief Report we consider a different, but related, problem, namely, the number of times a site is visited by a random flier.

Suppose that a random walk takes place on a d-dimensional lattice $\mathcal{L}$; let $\mathbf{s}$ be a site of $\mathcal{L}$, and let $P_k^{(d)}(\mathbf{s})$ be the probability that after k steps the walker is at $\mathbf{s}$. The mean value of visits to the site $\mathbf{s}$ after n steps is [5]

$$M_n^{(d)}(\mathbf{s})=\sum_{k=0}^{n}P_k^{(d)}(\mathbf{s}); \tag{1}$$

since derivation of Eq. (1) does not depend on the specific form of the walk [5], it holds also for Levy flights. In the following it will be assumed that $M_0(\mathbf{s})=P_0(\mathbf{s})=\delta_{\mathbf{s},0}$; the asymptotic value of $M_n^{(d)}$, denoted by $\mathcal{M}^{(d)}$, is defined as $\mathcal{M}^{(d)}=\lim_{n\to\infty}M_n^{(d)}$. It is known [6] that a random walk is transient if and only if $\Sigma_{k=0}^{\infty}P_k^{(d)}(\mathbf{s})<\infty$; in other words the existence of finite $\mathcal{M}^{(d)}$ implies that the walk is transient.

Levy flights have a wide range of applications (see, for instance, [7] and references therein) and, in particular, analysis of the number of times a site is visited can be relevant in those processes, such as random searches, in which it is important not just to determine what sites have been visited but how often they have been visited; examples of possible applications range from animal foraging [8] to exploration of visual space [9]. Moreover $M_n^{(d)}$ can be given the following interpretation, useful for possible applications: assume that particles undergoing a Levy flight are continuously generated at the origin; then, at time n, $M_n^{(d)}(\mathbf{s})\propto C_n^{(d)}(\mathbf{s})$, where $C_n^{(d)}(\mathbf{s})$ is the mean number of particles at site $\mathbf{s}\neq\mathbf{0}$ [10]. This property of $M_n^{(d)}$ has been used, in a model based on electron Brownian motion, to simulate distributions of emissivity of the supernova remnants [10].

II. INFINITE LATTICES

Consider first one-dimensional, infinite lattices; the probability of occupancy of site x after k steps is [11]

$$P_{k+1}^{(1)}(x)=\sum_{x=-\infty}^{\infty}p(x-y)P_k^{(1)}(y), \tag{2}$$

where $p(y)$ is the probability of having a displacement of y sites. In case of symmetric Levy flights the canonical representations of p and $P_k^{(1)}$ are [1,2]

$$p(y)=\frac{1}{\pi}\int_0^{\infty}\cos qy\exp(-cq^{\alpha})dq, \tag{3}$$

$$P_k^{(1)}(x)=\frac{1}{\pi}\int_0^{\infty}\cos qx\exp(-ckq^{\alpha})dq, \tag{4}$$

where $0<\alpha<2$ and c is a real number, which in the following will be set equal to 1 for simplicity [2]; a scaling relation holds between $P_k^{(1)}$ and p, namely, $P_k^{(1)}(x)=k^{-1/\alpha}p(xk^{-1/\alpha})$. If $\alpha=2$ Eqs. (3) and (4) yield the Gaussian distribution [1,6], whereas, if $\alpha>2$, P_k^1 fails to be a proper distribution not concentrated at a point [6]; therefore representations (3) and (4) are valid only in the interval $0<\alpha\leq 2$. More recently it has been shown that the analytic forms of p and $P_k^{(1)}$ can be given through a Fox function [12].

Application of (1) and of the scaling relation leads to $M_n^{(1)}(x)=\delta_{x,0}+\Sigma_{k=1}^{n}k^{-1/\alpha}p\left(\frac{x}{k^{1/\alpha}}\right)$, and in particular, recalling that $p(0)=(\pi\alpha)^{-1}\Gamma(1/\alpha)$ [2],

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$$M_n^{(1)}(0) = 1 + \frac{\Gamma(1/\alpha)}{\pi\alpha}\sum_{k=1}^{n} k^{-1/\alpha}, \tag{5}$$

with $\sum_{k=1}^{n} k^{-1/\alpha}$ converging to a finite value for $n \to \infty$ if and only if $\alpha < 1$ [13]; in this case

$$\mathcal{M}^{(1)}(0) = 1 + \frac{\Gamma(1/\alpha)}{\pi\alpha}\zeta(1/\alpha), \tag{6}$$

where ζ is the well known Riemann zeta function [13]. Thus Eqs. (5) and (6) show that the visit to site $x=0$ is a transient state if and only if $\alpha < 1$.

The trend of $M_n^{(1)}(0)$ as n increases can be computed by making use of the formulas related to the ζ function [13]; for $\alpha < 1$ the result is

$$M_n^{(1)}(0) = 1 + \frac{\Gamma(1/\alpha)}{\pi\alpha}\left(\zeta(1/\alpha) - \frac{\alpha}{1-\alpha} n^{(\alpha-1)/\alpha} + \frac{1}{\alpha}\int_n^{\infty} \frac{z-[z]}{z^{1/\alpha+1}} dz\right), \tag{7}$$

where $[z]$ is the integer part of z. Application of standard summation formulas [14] shows that, if $\alpha=1$, $M_n^{(1)}(0)$ grows logarithmically, whereas, if $1 < \alpha < 2$,

$$M_n^{(1)}(0) \sim \frac{\Gamma(1/\alpha)}{\pi(\alpha-1)} n^{(\alpha-1)/\alpha}, \tag{8}$$

as $n \to \infty$; finally in the case of a classical random walk ($\alpha \geq 2$), $M_n(0) = O(n^{1/2})$ [10]. Since flights are symmetric and start from 0, P_k is, for every k, an even function with a maximum in 0 [12] and hence $M_n^{(1)}(0) > M_n^{(1)}(x)$, for every n and for every $x \neq 0$; therefore, if $\alpha < 1$, $\mathcal{M}^{(1)}(x) < \infty$. A series expansion of Eq. (4) shows that

$$M_n^{(1)}(x) = M_n^{(1)}(0) - 1 + \sum_{l=1}^{\infty} (-1^l) \frac{\Gamma\left(\frac{2l+1}{\alpha}\right)}{\pi\alpha} \frac{x^{2l}}{(2l)!} \sum_{k=1}^{n} k^{-(2l+1)/\alpha}; \tag{9}$$

now for every $0 < \alpha < 2$ and every l, $2l+1/\alpha > 1$, and $\lim_{n\to\infty}\sum_{k=1}^{n} k^{-(2l+1)/\alpha} = \zeta((2l+1)/\alpha)$ is finite. Then $M_n^{(1)}(x) = O(M_n^{(1)}(0))$, that is, the last term on the right-hand side of Eq. (9) just takes into account the delay with which the flier reaches site x; in particular, if $\alpha > 1$, $M_n^{(1)}(0)$ diverges and $M_n^{(1)}(x) \sim M_n^{(1)}(0)$. In conclusion, a one-dimensional flight is transient if and only if $\alpha < 1$, a result that has been obtained in a somehow more complex way in [3].

Consider now a d-dimensional lattice, with $d \geq 2$, and assume the probabilities along the different coordinates to be independent; then Eq. (5) becomes $M_n^{(d)}(0) = 1 + \left(\frac{\Gamma(1/\alpha)}{\pi\alpha}\right)^d \sum_{k=1}^{n} k^{-d/\alpha}$. Note that, for $0 < \alpha < 2$ and $d \geq 2$, the condition $d/\alpha > 1$ holds and hence $\mathcal{M}^{(d)}(0) = 1 + \left(\frac{\Gamma(1/\alpha)}{\pi\alpha}\right)^d \zeta(d/\alpha)$ is finite; $M_n^{(d)}(0)$ as a function of n can be computed by using a method similar to the one-dimensional case, and the result is that the trend is given by $F(n) = \left(\frac{\Gamma(1/\alpha)}{\pi\alpha}\right)^d (n^{(\alpha-d)/\alpha} - 1) + O(n^{-d/\alpha})$. Finally it should be observed that the results for $M_n^{(1)}(x)$, $x \neq 0$, obtained above, can be extended in a straightforward way to multidimensional lattices. Thus Levy flights on lattices of dimensions higher than 1 are always transient; if $\alpha \geq 2$, $M_n^{(2)}(0) = O(\log(n))$ and, if $d > 2$, $M_n^{(d)}(0)$ converges to a finite value [10], and the walk is transient [6].

Note that, when $\alpha=1$, $M_n^{(1)}(0)$ has the same trend as $M_n^{(2)}(0)$ in the Gaussian regime, an instance of Levy flights increasing the effective dimension of the walk [11].

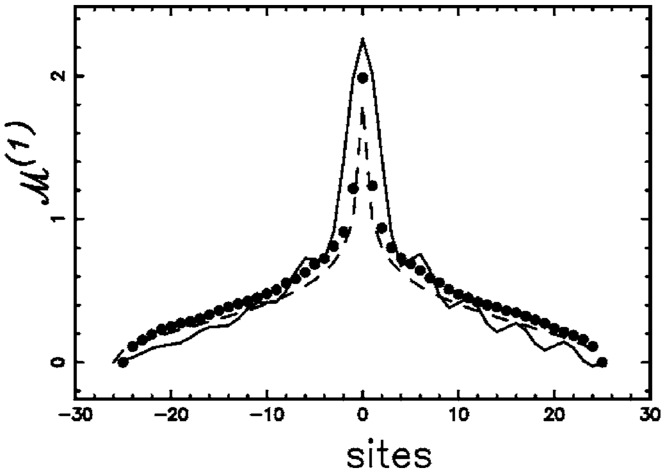

FIG. 1. Graphs of $\mathcal{M}^{(1)}$, in case of absorbing boundaries: points denote the Monte Carlo simulation, the dashed line the numerical method via Eqs. (10) and (1), and the full line the analytical solution [see Eq. (14)]. The parameters are $\alpha=0.8$, $L=51$; the Monte Carlo simulation comprises 10 000 trials, in Eq. (1) $n=2000$, and the index m in Eq. (14) ranges from 1 to 20.

III. FINITE INTERVALS WITH ABSORBING BOUNDARIES

In case of flights on a bounded set it is obvious that for reflecting boundaries $M_n^{(d)}$ diverges as n increases, since asymptotically $P_k^{(d)} \approx 1/|\mathcal{L}|$, where $|\mathcal{L}|$ is the number of sites [6], whereas if boundaries are absorbing $\mathcal{M}^{(d)}$ exists; here we shall consider just the case of one-dimensional lattices with absorbing boundaries. The map $M_n^{(1)}$ can be computed by means of numerical or analytical methods. In fact, Eq. (2) can be seen as a recursive method to compute $P_k^{(1)}$ and application of (1) provides the result; alternatively, one can use the diffusion approximation to derive an analytical formula. Both methods have been used here and their results have been compared with $\mathcal{S}(x)$, the "experimental" number of visits generated by a Monte Carlo simulation.

In a closed interval $[-a,a]$ Eq. (2) becomes

$$P_{k+1}^{(1)}(x) = \sum_{x=-a}^{a} p(x-y)P_k^{(1)}(y); \tag{10}$$

here, for reason of simplicity, instead of (3), we have used the transition probability, defined on integers y,

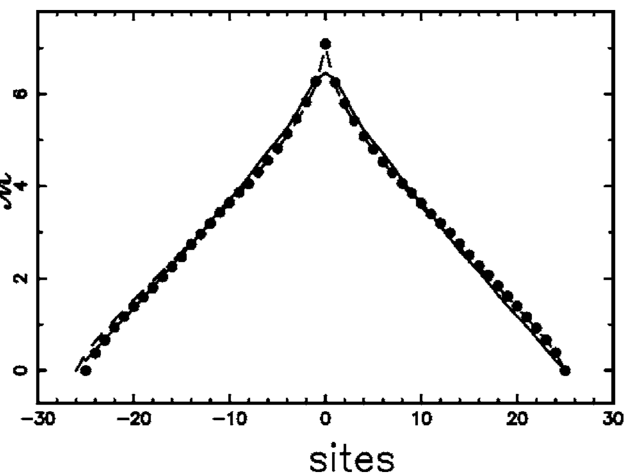

FIG. 2. The same as Fig. 1 but $\alpha=1.8$

$$p(y)=\frac{1}{Z}|y|^{-(\alpha+1)} \quad \text{if } y \neq 0, \tag{11}$$

and $p(0)=0$, Z being a normalizing constant. A similar form of p has been used in a work on the average time spent by flights in a closed interval [15]. In case of numerical calculations, obviously, the absolute length $|y|$ of a step must be truncated to some finite value: here $\max(|y|)=2a$, to allow flights to encompass the whole interval, and consequently $Z=\Sigma_{y=-2a}^{2a}|y|^{-(\alpha+1)}$, $y\neq 0$. Equation (11) provides a valid transition probability for any $\alpha>0$ and hence it can be used to model also classical Brownian motion; for $\alpha\rightarrow\infty$ the process becomes the simple symmetric walk. Note that by combining (2) and (1) a recursive formula for $M_n^{(1)}$ can be derived, namely, $M_{n+1}^{(1)}(x)=\Sigma_{x=-a}^{a}p(x-y)M_n^{(1)}(y)+\delta_{x,0}$; however, the separate use of Eqs. (2) and (1) is to be preferred, in that it also yields values of the probability distribution and this is useful to check the correctness of the results.

In the classical theory of a random walk the diffusion approximation allows us to replace $P_k^{(1)}(x)$ with the probability density function (PDF) $P^{(1)}(x,t)$, the solution of the diffusion equation [16]; analogously for Levy flights a superdiffusion equation can be derived (see, among others, [12,15,17]), whose solution is a series of eigenfunctions f_k of the operator $\mathcal{D}_\alpha$ [15]. Setting $P^{(1)}(x,0)=\delta(x-0)$, the PDF is $P^{(1)}(x,t)=\Sigma f_m(0)f_m(x)\exp(\lambda_m t)$. Define, in analogy with the discrete case,

$$M^{(1)}(x,t)=\int_0^t P^{(1)}(x,\tau)d\tau; \tag{12}$$

then $M^{(1)}(x,t)=\Sigma_{m=1}^{\infty}\lambda_m^{-1}f_m(0)f_m(x)[\exp(\lambda_m t)-1]$ where λ_k are the eigenvalues of $\mathcal{D}_\alpha$; obviously, $\lambda_k<0$, for all k, and the asymptotic formula is $\mathcal{M}^{(1)}(x)=\Sigma_{m=1}^{\infty}|\lambda_m|^{-1}f_m(0)f_m(x)$.

In [17] a solution $P^{(1)}(x,t)$ of the superdiffusion equation has been presented that, for symmetric flights, is

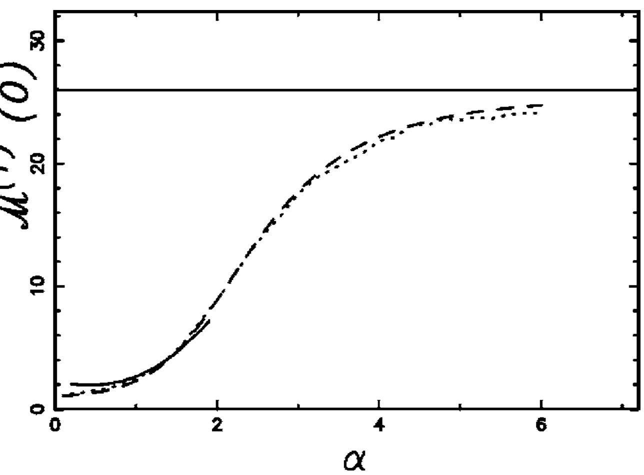

FIG. 3. Graphs of $\mathcal{M}^{(1)}(0)$, as a function of α, starting from $\alpha=0.2$, in case of absorbing boundaries. Points represent the Monte Carlo simulation, the dashed line the numerical solution of (10) and (1), and the continuous line results from Eq. (14), with $\alpha<2$. The horizontal line is the result for $\alpha=\infty$.

$$P^{(1)}(x,t)=\frac{2}{L}\sum_{m=1}^{\infty}\exp[-D_\alpha(\pi m/L)^\alpha t] \times \sin\left(\frac{m\pi(x+a)}{L}\right)\sin\left(\frac{m\pi a}{L}\right); \tag{13}$$

here $L=2a$ is the length of the interval and D_α the diffusion coefficient; application of Eq. (12) and (13), with $t\rightarrow\infty$, provides an explicit form for $\mathcal{M}^{(1)}(x)$,

$$\mathcal{M}^{(1)}(x)=\frac{2}{L}\sum_{m=1}^{\infty}\frac{L^\alpha}{(m\pi)^\alpha D_\alpha}\sin\left(\frac{m\pi(x+a)}{L}\right)\sin\left(\frac{m\pi a}{L}\right). \tag{14}$$

Calculations of $\mathcal{M}^{(1)}(x)$ from Eq. (14) need the numerical value of the diffusion coefficient D_α, and it can be derived from the average time T a flier spends in the interval, related to D_α by the formula [17]

$$T=\frac{4}{\pi D_\alpha}\left(\frac{L}{\pi}\right)^\alpha\sum_{m=1}^{\infty}\frac{(-1)^m}{(2m+1)^{\alpha+1}}; \tag{15}$$

since T is defined as $T=\int_{-a}^{a}dx\int_0^\infty P(x,t)dt=\int_{-a}^{a}\mathcal{M}^{(1)}(x)dx$ the approximation $T\approx\Sigma_{x=-a}^{a}\mathcal{S}(x)$ can be used to obtain the numerical value of D_α.

Figures 1 and 2 show $\mathcal{M}^{(1)}$ for $\alpha=0.8$ and 1.8, respectively. It can be seen that the graph of $\mathcal{M}$ tends to a triangular shape as α increases; indeed for a simple symmetric random walk $[\alpha=\infty,\ p(x)=1/2\,\delta_{|x|,1}]$, $\mathcal{M}^{(1)}(x)=a-|x|$ [10].

Figure 3 presents the graph of $\mathcal{M}^{(1)}(0)$ as a function of α; note that the inflection point of the curve occurs at $\alpha=2$, that is, at the boundary between Levy flights and classical random walks. In other words, $\mathcal{M}^{(1)}(0)$ shows a "phase transition" from Levy flights, characterized by a small number of visits, to the Gaussian regime where visits are more frequent.

IV. CONCLUSION

The results of this Brief Report clarify how the mean number of times a site is visited by a random flier depends on the dimensionality of the lattice, the value of α, and the boundary conditions. In particular, it has been shown that unrestricted Levy flights are always transient, but for the unidimensional case with $\alpha \geq 1$, and that restricted flights are transient if the boundaries are absorbing. In the last case computations show that the direct numerical method agrees very closely with experimental data generated by a Monte Carlo simulation, whereas the agreement is worse for Eq. (14), especially when α is small (see Figs. 1 and 3); this is not surprising, since Eq. (10) deals directly with discrete variables, whereas Eq. (14) results from the diffusion approximation. On the other hand, obviously, Eq. (14) provides a more general, analytical formula for $\mathcal{M}^{(1)}$ and not just a set of numerical values.

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